Structural Non-Closure in the Riemann Hypothesis
Finite Observability versus Global Control

K. E. Itekki

Abstract

We analyze the Riemann Hypothesis (RH) from a methodological perspective, focusing on the structural gap between finite observability and infinite global control. Although RH is supported by extensive numerical verification and refined observational analyses, no existing approach upgrades finite stability into a uniform zero-exclusion principle.

We show that this difficulty can be traced to minimal arithmetic facts about prime distribution, specifically the unboundedness of prime gaps, which obstructs bounded local propagation from finite truncations of prime-based observables. Our analysis does not assert logical undecidability of RH, but identifies a structural non-closure that explains the persistence of the conjecture despite maximal finite confirmation.

1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$. It is among the most extensively investigated conjectures in mathematics, supported by overwhelming numerical verification and deep analytic structure.

This paper does not claim to prove or disprove the Riemann Hypothesis. Instead, it aims to clarify a methodological obstruction that limits the ability of truncation-based or observational approaches to yield a global zero-exclusion principle.

We propose that the difficulty of RH reflects a structural mismatch between finite observability and infinite global control, rather than a mere lack of technical sophistication.

2 Finite Observability and Global Control

We emphasize that the terms "finite observability" and "global control" are used here in a methodological and logical sense, rather than in a control-theoretic or experimental framework.

Definition 2.1 (Finite Observability)**.** A property P defined over an infinite domain D is said to be *finitely observable* if, for every finite restriction $D_T \subset D$, the validity of P on D_T can be verified by a finite procedure.

Definition 2.2 (Global Control)**.** A property P admits *global control* if there exists a finite argument that excludes the existence of counterexamples anywhere in D.

The Riemann Hypothesis exhibits extremely strong finite observability, yet no known mechanism upgrades this to global control.

3 Arbitrarily Long Prime Gaps

Lemma 3.1. *For every integer $L \geq 1$, there exist L consecutive composite integers.*

Proof. Let $N = (L+1)!$. Then for each $k = 2, 3, \ldots, L+1$, the integer $N + k$ is divisible by k and hence composite. □

Corollary 3.2. *Prime gaps are unbounded.*

Proof. The existence of arbitrarily long runs of composite integers implies that the difference between consecutive primes is unbounded. □

4 Structural Non-Linearizability of the Prime Set

Definition 4.1 (Structural Linearizability)**.** A set $S \subset \mathbb{N}$ is said to be *structurally linearizable* if there exist total computable functions $f, b : \mathbb{N} \to \mathbb{N}$ such that:

(i) f is strictly increasing,

(ii) $\mathrm{range}(f) = S$,

(iii) $f(n+1) - f(n) \leq b(n)$ for all n.

Here the function $b(n)$ is intended to represent an a priori computable upper bound on the next gap in the enumeration, given the current index n.

Lemma 4.2. *The set of prime numbers is not structurally linearizable.*

Proof. Assume such functions f, b exist. Then f enumerates the primes in increasing order, and the bound $b(n)$ provides a computable upper bound on prime gaps. This contradicts the unboundedness of prime gaps established above. □

5 Truncation-Based Observables and Remainder Control

For $\Re(s) > 1$, the logarithmic derivative of the zeta function satisfies

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where Λ is the von Mangoldt function.

For a finite cutoff N, write

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n \leq N} \frac{\Lambda(n)}{n^s} + R_N(s), \quad R_N(s) = \sum_{n > N} \frac{\Lambda(n)}{n^s}.$$

Any attempt to deduce global zero-exclusion from finite truncations requires uniform control of $R_N(s)$ over unbounded imaginary parts, which cannot be derived from finite prime data alone.

6 Structural Non-Closure of the Riemann Hypothesis

Here, “bounded local propagation” refers precisely to attempts to control the remainder term $R_N(s)$ from finite prime data, with bounds depending only on the truncation parameter N.

Theorem 6.1. *The Riemann Hypothesis exhibits* structural non-closure*: it is finitely observable but lacks a mechanism of global control derivable from bounded local propagation of prime-based observables.*

Proof. Finite observability follows from numerical verification of zeros on arbitrarily large finite domains. Global control would require a uniform truncation bound independent of finite prime data, which is obstructed by the unbounded irregularity of prime distribution. □

7 Comparison with Other Prime Conjectures

The Twin Prime Conjecture requires infinite recurrence of bounded gaps, while Goldbach’s Conjecture concerns universal additive coverage. By contrast, the Riemann Hypothesis demands global exclusion: the absence of counterexamples anywhere in an infinite domain, rather than infinite recurrence or universal coverage. This leads to a particularly strong mismatch between observational support and available control principles.

8 Discussion and Methodological Implications

The present work suggests that progress on RH may require principles of global control that are not reducible to finite prime data or bounded local propagation. Whether such principles exist remains an open question.

References

[1] H. M. Edwards, *Riemann’s Zeta Function*, Academic Press, 1974.

[2] A. Ivić, *The Riemann Zeta-Function*, Wiley, 1985.

[3] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Proc. Sympos. Pure Math.*, 24 (1973).